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Common Fixed Points and Best Approximation

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We prove a common fixed-point theorem generalizing results of Dotson and Habiniak. Using this result, we extend, generalize, and unify well known results on fixed points and common fixed points of best approximation.

1. INTRODUCTION

Let X be a normed linear space, D and M subsets of X, I and T selfmaps of X, $F(I, T)$ the set of common fixed points of I and T, and $F(I)$ the set of fixed points of I. I and T commute on D if $ITx = TIx$ for all $x \in D$. T is I-nonexpansive [resp. nonexpansive] on D if $\|Tx - Ty\| \leqslant \|Ix - Iy\|$ [resp. $\|Tx - Ty\| \leqslant \|x - y\|$] for every x, $y \in D$. T is I-contraction on D if $\|Tx - Ty\| \leqslant k\|Ix - Iy\|$ for every x, $y \in D$ and some $k \in [0, 1)$. D is q-starshaped if there exists $q \in D$ such that $(1-k)q + kx \in D$ for all $x \in D$ and all $k \in (0, 1)$. D is convex if it is q-starshaped for every $q \in D$. Let $B_M(p) := \{x \in M\colon \|x - p\| = \delta(p, M)\}$ be the set of best M-approximants to $p \in X$, where $\delta(p, M) = \inf_{z \in M} \|z - p\|$, and let $C_M^I(p) := \{x \in M\colon Ix \in B_M(p)\}$.

Brosowski [1] proved that if T is nonexpansive with $p \in F(T)$, $T(M) \subset M$ and $B_M(p)$ is nonempty, compact and convex, then T has a fixed point in $B_M(p)$. Subrahmanyam [11] replaced the requirement that $B_M(p)$ is nonempty by the assumption that M is a finite-dimensional subspace of X. Singh [8] noted that Brosowski's result remains true if $B_M(p)$ is only q-starshaped. Singh [9] noted that the nonexpansiveness of T on $B_M(p) \cup \{p\}$ is enough for his earlier result. Hicks and Humphries [4] noted that Singh's first result remains true if $T(M) \subset M$ is replaced by $T(\partial M) \subset M$, where ∂M is the boundary of M in X. Smoluk [10] noted that the finite-dimensionality of M in Subrahmanyam's result can be replaced by the assumptions that T is linear and that $\mathrm{Cl}(T(D))$ is compact for every bounded subset D of M. Habiniak [3] removed the linearity of T from Smoluk's result. Sahab, Khan and Sessa [7] unified and generalized the result of Hicks and Humphries and the results of Singh by the following:

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THEOREM 1.1 *Let X be a normed linear space, I and T selfmaps of X with $p \in F(I, T)$, $M \subset X$ with $T(\partial M) \subset M$, and $q \in F(I)$. If $D := B_M(p)$ is compact and q-starshaped, $I(D) = D$, I is continuous and linear on D, I and T are commuting on D and T is I-nonexpansive on $D \cup \{p\}$, then I and T have a common fixed point in D.*

In section 2, we prove a common fixed-point generalization of Habiniak's extension [3] of a fixed-point result of Dotson [2]. We use this result to extend, generalize and unify the above results on fixed points and common fixed points of best approximation. The first result in section 3 extends Theorem 1.1. The second result generalizes the above results of Singh and the result of Hicks and Humphries. The first result in section 4 extends Habiniak's result. The second result shows that a common fixed point of best approximation exists. It is a generalization of Smoluk's result and, in a certain case, of our extension of Habiniak's result.

2. COMMON FIXED-POINTS THEOREMS

THEOREM 2.1. *Let D be a closed subset of a metric space X and let I and T be selfmaps of D with $T(D) \subset I(D)$.* If $\mathrm{Cl}(T(D))$ *is complete, I is continuous, I and T are commuting and T is I-contraction, then I and T have a unique common fixed point.*

Theorem 2.1 is an extension of a common fixed-point result of Jungck [5]. The proof is a slight modification of Jungck's proof.

THEOREM 2.2. *Let D be a closed subset of a normed linear space X, I and T selfmaps of D with $T(D) \subset I(D)$, and $q \in F(I)$. If D is q-starshaped,* $\mathrm{Cl}(T(D))$ *is compact, I is continuous and linear, I and T are commuting and T is I-nonexpansive, then I and T have a common fixed point.*

Proof. For each positive integer n, let $k_n = n/n+1$ and define $T_n: D \to D$ by $T_n x = (1 - k_n) q + k_n Tx$ for each $x \in D$. Since D is q-starshaped, I is linear and $T(D) \subset I(D)$, then $I(D)$ is q-starshaped and, therefore, $T_n(D) \subset I(D)$ for each n. To show that I and T_n commute, note that I and T commute and that I is linear, and hence

$$\begin{aligned} T_n Ix &= (1 - k_n) q + k_n TIx \\ &= (1 - k_n) Iq + k_n ITx = I[(1 - k_n) q + k_n Tx] = IT_n x. \end{aligned}$$

for all $x \in D$. Since T is I-nonexpansive, we have

$$\|T_n x - T_n y\| = k_n \|Tx - Ty\| \leqslant k_n \|Ix - Iy\|$$

for every $x, y \in D$. Therefore each T_n is I-contraction. Since $\mathrm{Cl}(T(D))$ is compact, each $\mathrm{Cl}(T_n(D))$ is compact. It follows from the continuity of I and Theorem 2.1 that $F(I, T_n) = \{x_n\}$ for some $x_n \in D$. Since $\{Tx_n\}$ is a sequence in $\mathrm{Cl}(T(D))$, there exists a subsequence $\{Tx_{n_j}\}$ with $Tx_{n_j} \to x_0 \in \mathrm{Cl}(T(D))$. Since

$$x_{n_j} = T_{n_j} x_{n_j} = (1 - k_{n_j}) q + k_{n_j} T x_{n_j} \to x_0,$$

the continuity of I and of T imply $x_0 \in F(I, T)$.

Corollary 2.3 [3]. *Let D be a closed q-starshaped subset of a normed linear space X and T a nonexpansive selfmap of D. If $\mathrm{Cl}(T(D))$ is compact, then T has a fixed point.*

Dotson [2] proved Corollary 2.3 in case D is compact.

Corollary 2.4 [6]. *Let X be a normed linear space, I and T selfmaps of X with $p \in F(I, T)$, $M \subset X$ with $I(M) \subset M$ and $T(M) \subset M$, and $I^2 = I$ on X. If $D := C^I_M(p)$ is compact, $q \in I(D)$, $(1-k)q + kx \in D$ for all $x \in D$ and all $k \in (0, 1)$, $T(D) \subset I(D)$, I is continuous and linear on D, I and T are commuting on X and T is I-nonexpansive on X, then I and T have a common fixed point in $B_M(p)$.*

Proof. Note that $I^2 = I$ on X implies $q \in F(I)$ and, since $I(D) \subset B_M(p)$, we have $I(D) \subset D$. Therefore D is q-starshaped and $T(D) \subset I(D) \subset D$.

If I is the identity map on M, Corollary 2.4 is a special case of Singh's first result.

3. Applications to Best Approximations

Let $D^I_M(p) := B_M(p) \cap C^I_M(p)$ where $B_M(p)$ and $C^I_M(p)$ are as in section 1. The proof of the following proposition is straightforward:

Proposition 3.1. *Let X be a normed linear space, I and T selfmaps of X, $M \subset X$ and $p \in X$.*

(i) *$B_M(p) = C^I_M(p) = D^I_M(p)$ whenever I is the identity map on M.*

(ii) *$B_M(p) \cap F(I, T) = C^I_M(p) \cap F(I, T) = D^I_M(p) \cap F(I, T)$.*

(iii) *If $I(B_M(p)) \subset B_M(p)$, then $B_M(p) \subset C^I_M(p)$ and, hence, $D^I_M(p) = B_M(p)$.*

(iv) *If $I(C^I_M(p)) \subset C^I_M(p)$, then $I(D^I_M(p)) \subset I(C^I_M(p)) \subset D^I_M(p)$.*

THEOREM 3.2. *Let X be a normed linear space, I and T selfmaps of X with $p \in F(I, T)$, $M \subset X$ with $T(\partial M \cap M) \subset M$, and $q \in F(I)$. If $D := D_M^I(p)$ is closed and q-starshaped,* $\mathrm{Cl}(T(D))$ *is compact, $I(D) = D$, I is continuous and linear on D, I and T are commuting on D and T is I-nonexpansive on $D \cup \{p\}$, then I and T have a common fixed point in $B_M(p)$.*

Proof. Let $x \in D$. Then $\|(1-k)x + kp - p\| < \delta(p, M)$ for all $k \in (0, 1)$. Thus $\{(1-k)x + kp: k \in (0, 1)\} \cap M = \varnothing$ and so $x \in \partial M \cap M$. Since $T(\partial M \cap M) \subset M$, then $Tx \in M$. Since $Ix \in B_M(p)$ and T is I-nonexpansive on $D \cup \{p\}$, we have

$$\|Tx - p\| = \|Tx - Tp\| \leqslant \|Ix - Ip\| = \|Ix - p\| = \delta(p, M)$$

and hence $Tx \in B_M(p)$. Since I and T commute on D, then

$$\|ITx - p\| = \|TIx - Tp\| \leqslant \|I^2x - Ip\| = \delta(p, M)$$

and so $Tx \in C_M^I(p)$. Therefore $Tx \in D$ and, hence, $T(D) \subset I(D) = D$. Now the result follows from Theorem 2.2.

The following variant of Theorem 3.2 unifies and generalizes the results of Singh and the result of Hicks and Humphries which were mentioned in section 1.

THEOREM 3.3. *Let X be a normed linear space, I and T selfmaps of X with $p \in F(I, T)$, $M \subset X$ with $T(\partial M \cap M) \subset I(M) \subset M$, and $q \in F(I)$. If $D := D_M^I(p)$ is closed and q-starshaped,* $\mathrm{Cl}(T(D))$ *is compact, $I(M) \cap D \subset I(D) \subset D$, I is continuous and linear on D, I and T are commuting on D and T is I-nonexpansive on $D \cup \{p\}$, then I and T have a common fixed point in $B_M(p)$.*

Proof. Let $x \in D$. We can show, as in the proof of Theorem 3.2, that $Tx \in D$. Since $T(D) \subset I(M)$, there exists $z \in M$ such that $Tx = Iz$. Thus $z \in C_M^I(p)$ and, hence, $T(D) \subset I(C_M^I(p)) \subset B_M(p)$. This shows that $T(D) \subset I(M) \cap D \subset I(D) \subset D$. Now the result follows from Theorem 2.2.

Remark. Proposition 3.1 (iii, iv) shows that Theorems 3.2 and 3.3 hold for $D := B_M(p)$ [resp. $D := C_M^I(p)$]. Therefore Theorem 1.1 is a special case of Theorem 3.2.

4. FURTHER APPLICATIONS TO BEST APPROXIMATIONS

Let X be a normed linear space and let $\mathscr{C}_0$ denotes the class of closed convex subsets of X containing 0. For $M \in \mathscr{C}_0$ and $p \in X$, let $M_p = \{x \in M: \|x\| \leqslant 2\|p\|\}$. Clearly $B_M(p) \subset M_p \in \mathscr{C}_0$. The following result extends Habiniak's result [3].

THEOREM 4.1. *Let X be a normed linear space, T a selfmap of X with $p \in F(T)$, and $M \in \mathscr{C}_0$ with $T(M_p) \subset M$. If T is nonexpansive on $M_p \cup \{p\}$* and $\mathrm{Cl}(T(M_p))$ *is compact, then*

(i) $B_M(p)$ *is nonempty, closed and convex,*

(ii) $T(B_M(p)) \subset B_M(p)$, *and*

(iii) *T has a fixed point in $B_M(p)$.*

Proof. We may assume that $p \notin M$. If $x \in M \backslash M_p$, then $\|x\| > 2\,\|p\|$ and, so

$$\|x-p\| \geqslant \|x\| - \|p\| > \|p\| \geqslant \delta(p, M).$$

Thus $\alpha := \delta(p, M_p) = \delta(p, M) \leqslant \|p\|$. Since $\mathrm{Cl}(T(M_p))$ is compact and the norm is continuous, there exists $z \in \mathrm{Cl}(T(M_p))$ so that $\beta := \delta(p, \mathrm{Cl}(T(M_p))) = \|z-p\|$. Hence

$$\alpha \leqslant \beta \leqslant \delta(p, T(M_p)) \leqslant \|Tx-p\| \leqslant \|x-p\|$$

for all $x \in M_p$. This shows that $\alpha = \beta$ and, hence, (i) and (ii) follow. Since $\mathrm{Cl}(T(B_M(p)))$ is compact, (iii) follows from Corollary 2.3 with $D = B_M(p)$.

THEOREM 4.2. *Let X be a normed linear space, I and T selfmaps of X with $p \in F(I, T)$, and $M \in \mathscr{C}_0$ with $T(M_p) \subset I(M) \subset M$. Suppose that I is linear and nonexpansive on M_p, $\|Ix-p\| = \|x-p\|$ for all $x \in M$, I and T are commuting on M_p, T is I-nonexpansive on $M_p \cup \{p\}$, and that one of the following two conditions is satisfied:*

(a) $\mathrm{Cl}(I(M_p))$ *is compact,*

(b) $\mathrm{Cl}(T(M_p))$ *is compact and T is linear on M_p.*

Then

(i) $B_M(p)$ *is nonempty, closed and convex,*

(ii) $T(B_M(p)) \subset I(B_M(p)) \subset B_M(p)$, *and*

(iii) *I and T have a common fixed point in $B_M(p)$.*

Proof. Note that both I an T are nonexpansive on $M_p \cup \{p\}$ and that T satisfies the hypotheses of Theorem 4.1. Then, for both (a) and (b), (i) holds, $I(B_M(p)) \subset B_M(p)$ and $T(B_M(p)) \subset B_M(p)$. To obtain (ii), let $y \in T(B_M(p))$. Since $T(M_p) \subset I(M)$, there exist $z \in B_M(p)$ and $w \in M$ such that $y = Tz = Iw$. Since T is I-nonexpansive on $M_p \cup \{p\}$ and $\|Ix-p\| = \|x-p\|$ for all $x \in M$, then

$$\|Iw-p\| = \|Tz-p\| \leqslant \|Iz-p\| = \|z-p\| = \delta(p, M).$$

Thus $w \in C_M^I(p) = B_M(p)$ and, hence, (ii) holds. To obtain (iii), note that if (a) is satisfied, then I has a fixed point $q \in B_M(p)$ by Theorem 4.1. Now suppose that (b) is satisfied. By the linearity [resp. continuity] of T, $D_0 := T(B_M(p))$ is convex [resp. compact]. Hence $D_1 := \mathrm{Cl}(I(D_0))$ is convex and compact. Since I and T commute on $B_M(p)$, we have $I(D_0) \subset D_0$ and, by the continuity of I on $B_M(p)$, $I(D_1) \subset D_1$. It follows from Corollary 2.3 that I has a fixed point $q \in D_1 \subset B_M(p)$. Now (iii) follows from Theorem 2.2 with $D = B_M(p)$.

Remark. Let I be the identity map on M_p in Theorem 4.2. Then

(i) Smoluk's result is a special case of Theorem 4.2 (b), and

(ii) if M_p is compact, Theorem 4.1 is Theorem 4.2 (a).

QUESTION. *Can we replace conditions* (a) *and* (b) *of Theorem* 4.2 *by* "$\mathrm{Cl}(T(M_p))$ *is compact*"?

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